CSE 250
Data Structures

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212 Capen Hall

Day 04
Runtime Analysis
Textbook Ch. 7.3-7.4

Announcements

- Dr. Kennedy will be giving lecture on Friday and Monday
- PA 0 is due Friday
- Start PA 1 early!

From Lecture 01...

Option 1
- Very fast Prepend, Get First
- Very slow Get Nth

Option 2
- Very fast Get Nth, Get First
- Very slow Prepend

Option 3
- Very fast Get Nth, Get First
- Occasionally slow Prepend

From Lecture 01...

Option 1

- Very fast Prepend, Get First
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What is fast? slow?

Attempt #1: Wall-clock time?

- What is fast?
  - 10s? 100ms? 10ns?
  - …it depends on the task
- Algorithm vs Implemtation
  - Compare Grace Hopper's implementation to yours
- What machine are you running on?
  - Your old laptop? A lab machine? The newest, shiniest processor?
- What bottlenecks exist? CPU vs IO vs Memory vs Network…

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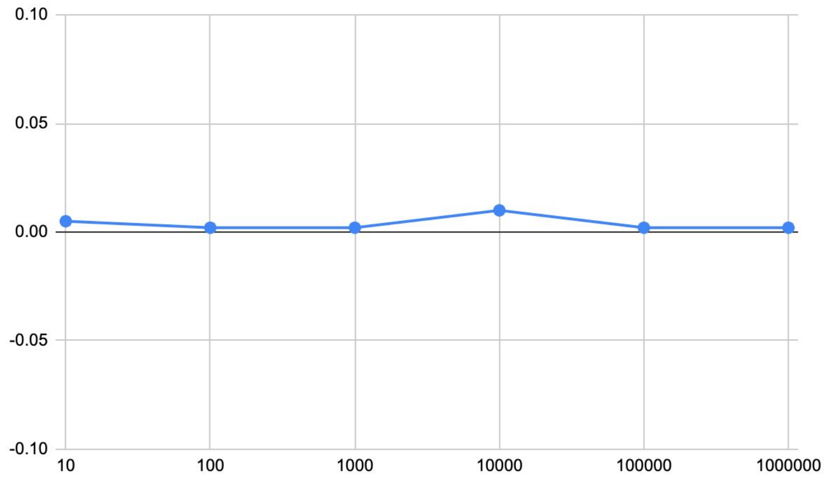
Wall-clock time is not terribly useful…

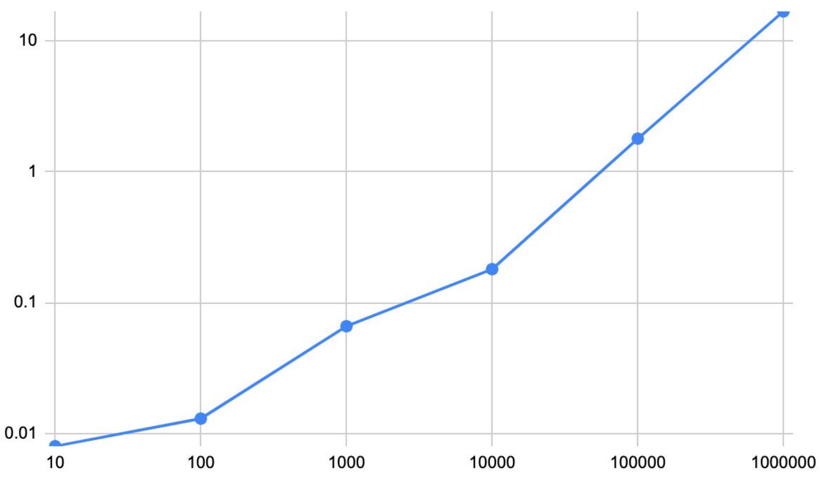
Let’s do a quick demo…

Comparing Random Access for Array vs List

Array



List



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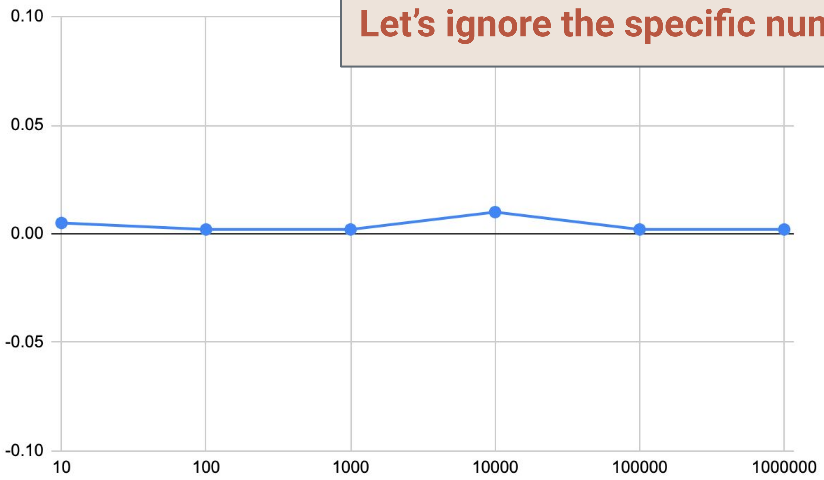

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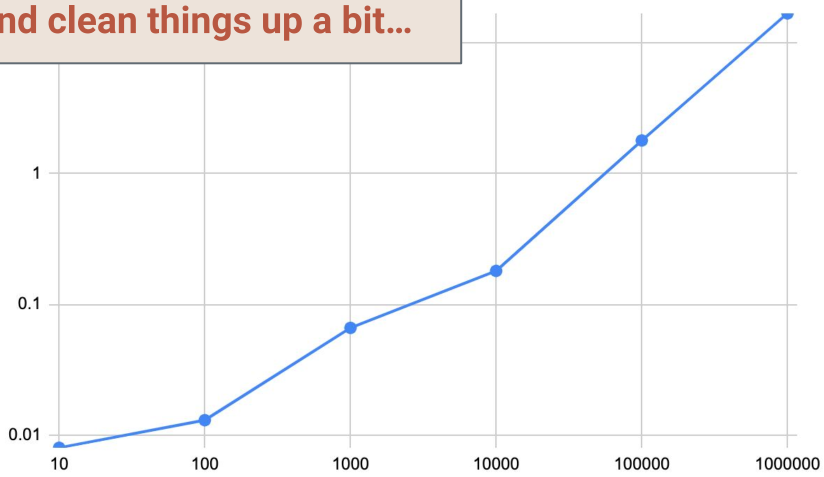
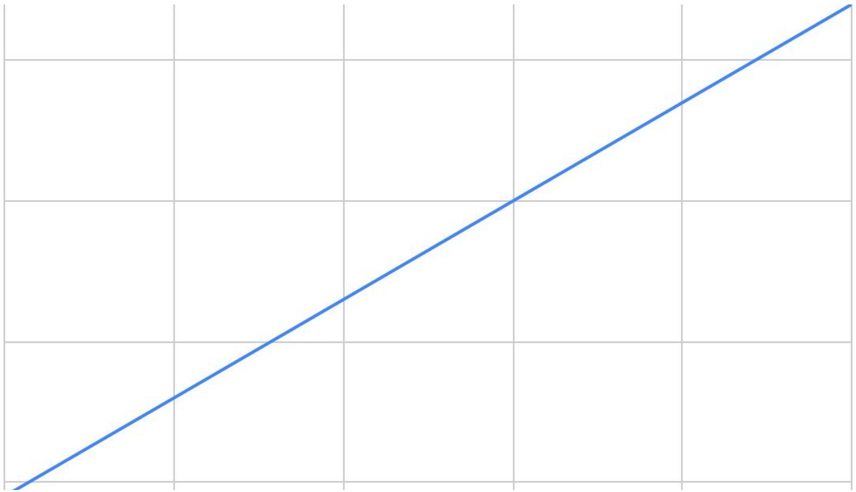


Let's ignore the specific numbers and clean things up a bit…

Comparing Random Access for Array vs List

Array

List



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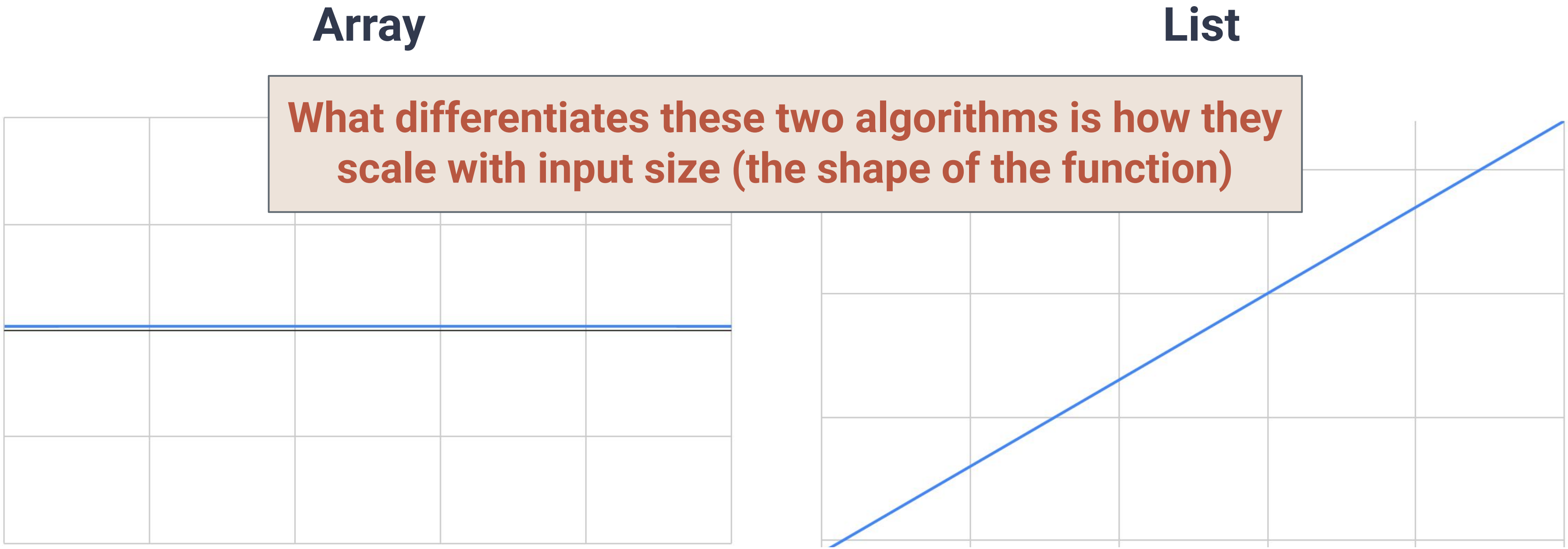

What differentiates these two algorithms is how they scale with input size (the shape of the function)

When is an algorithm “fast”?

- To give a useful solution, we should take “scale” into account
  - How does the runtime change as we change the size of the input (number of users, records, pixels, elements, etc)
- Don’t think in terms of wall-time, think in terms of “number of steps”

Scaling Examples

- “Five steps plus Ten steps per user”
- “Ten steps per network connection. Each node has connections to 1% of the other nodes in the system”
- “Seven steps for every possible pair of elements
- “For each user, Ten steps, plus Three steps per post”

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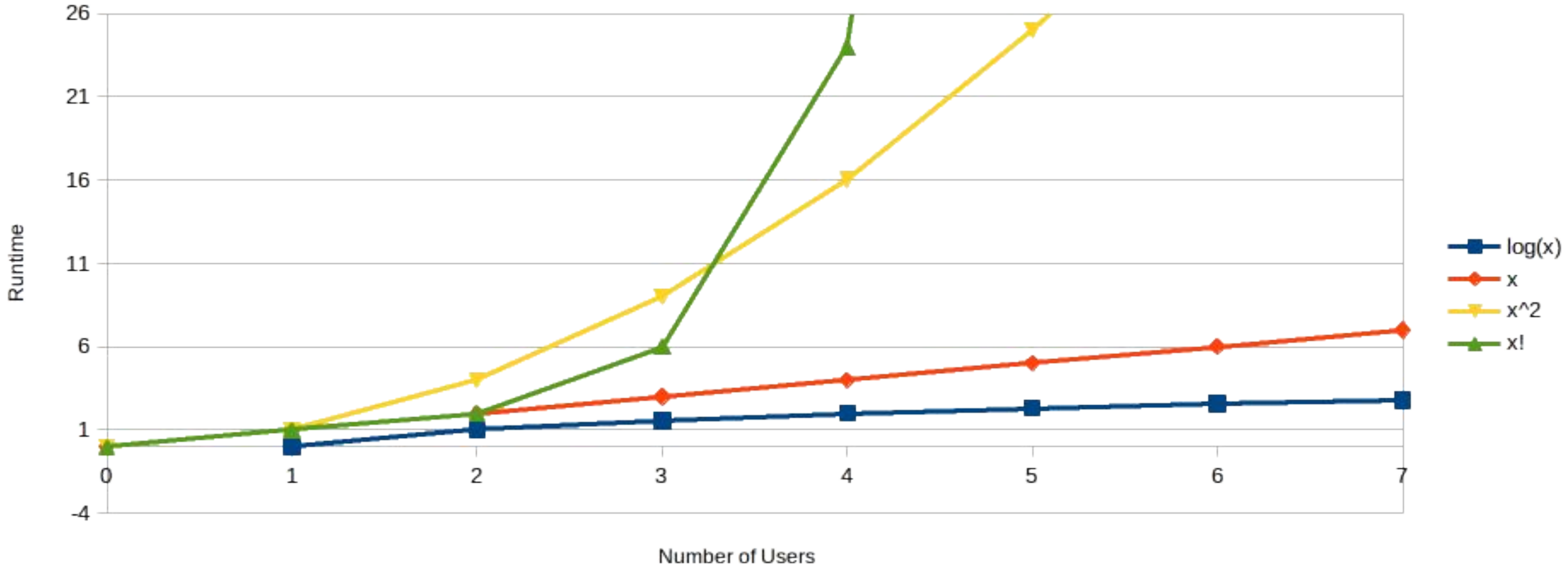
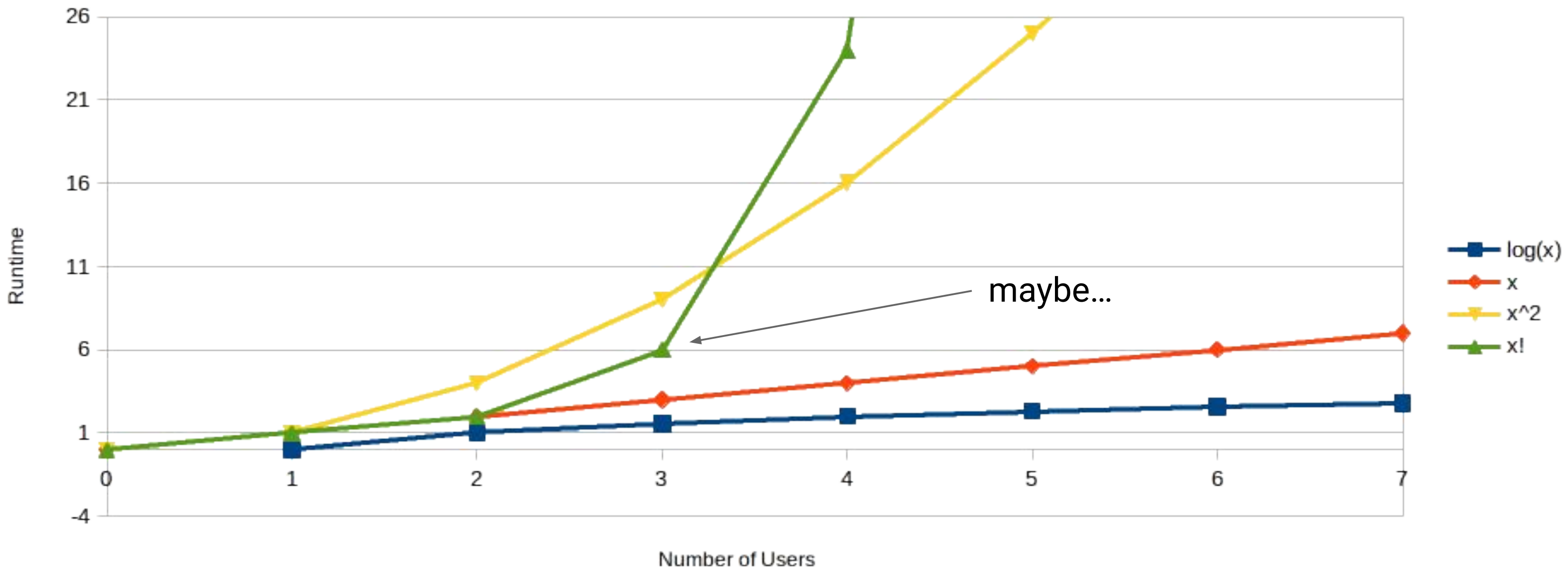
- “For each user, Ten steps, plus Three steps per post”
  - **|Users| x (10 + 3 x |Posts|)**

Runtime as a Function



Would you consider an algorithm that takes |Users|! number of steps?

Runtime as a Function



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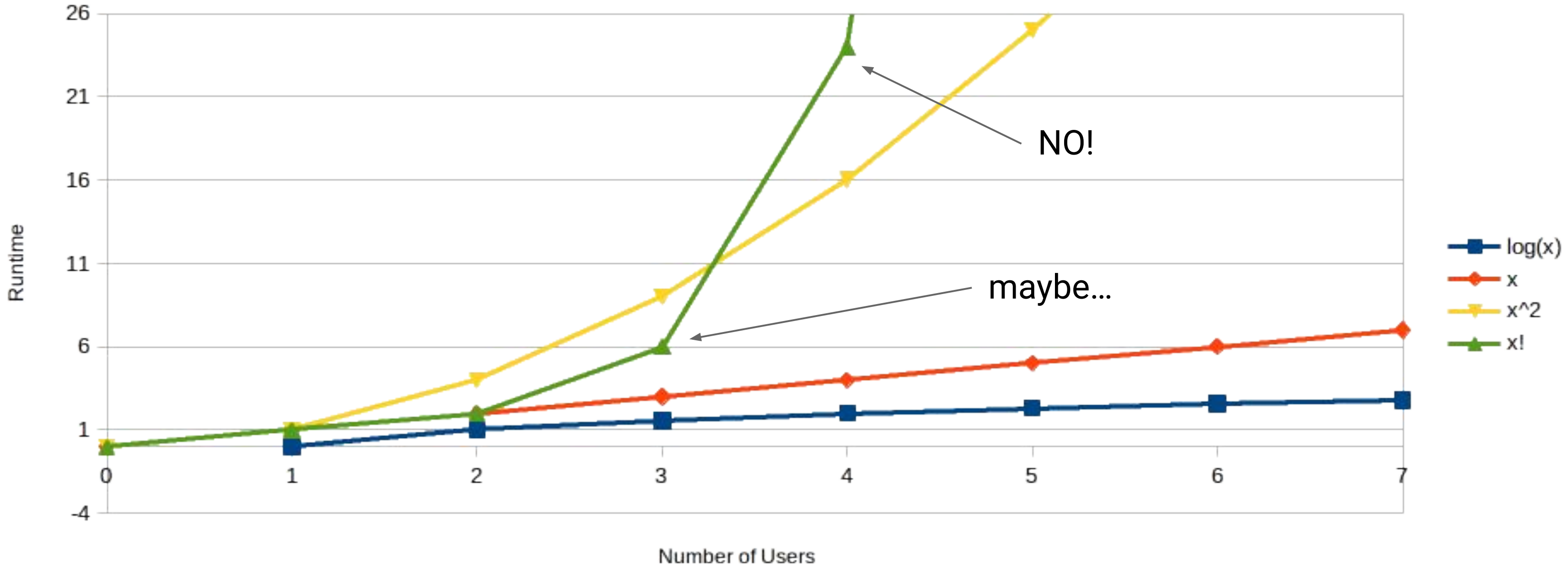

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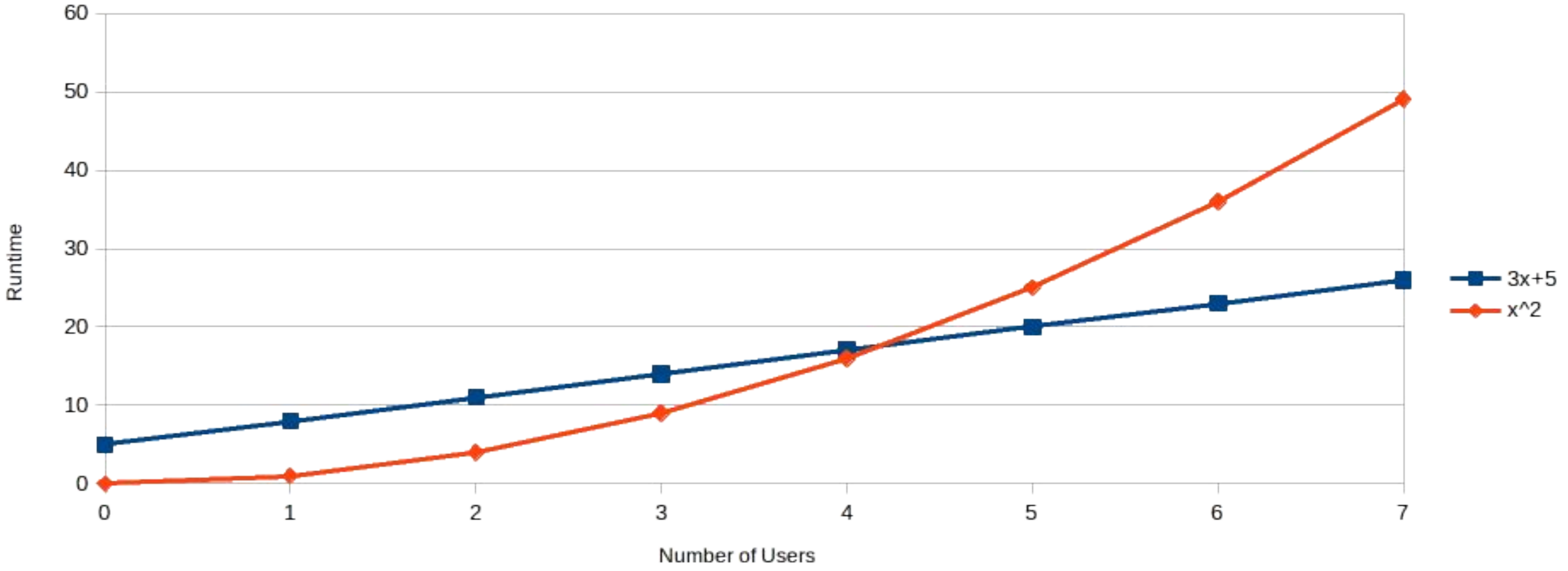

Which is better? 3x|Users|+5 or $|Users|^2$

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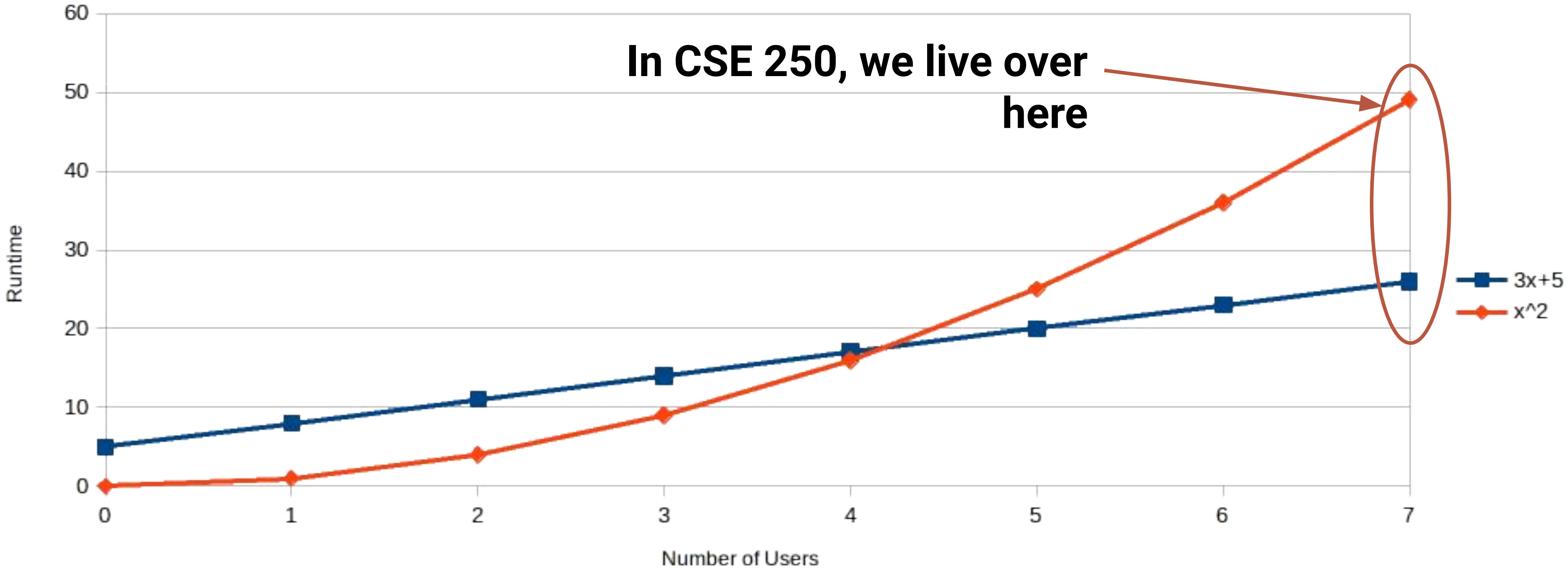

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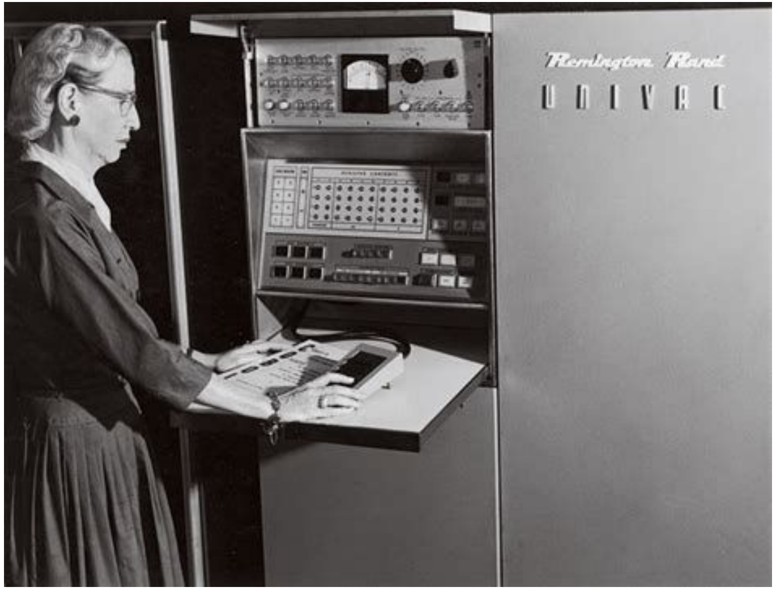
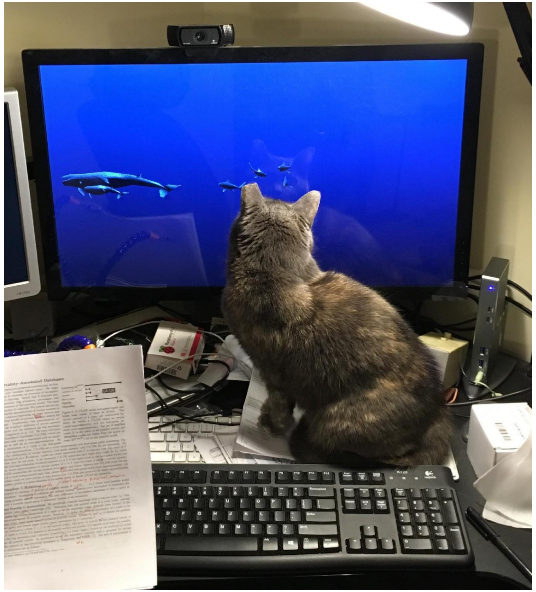
Goal: Ignore implementation details



Seasoned Pro Implementation

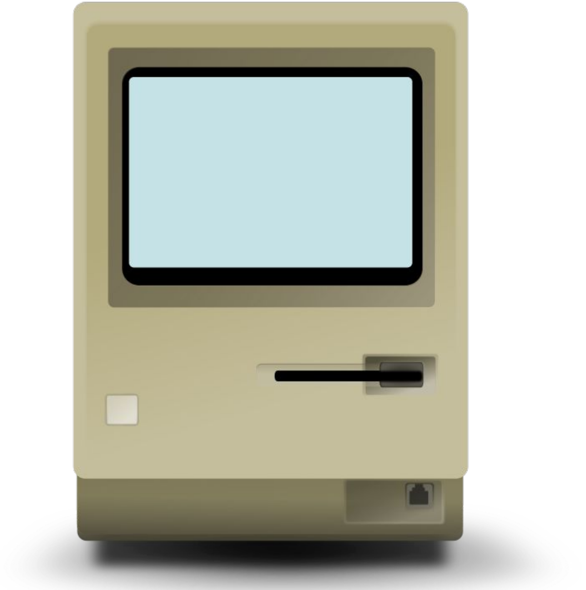
vs



Error 23: Cat on Keyboard

Goal: Ignore execution environment

vs



Intel i9

Motorola 68000

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Goal: Judge the Algorithm Itself

- How fast is a step? Don’t care
  - Only count number of steps
- Can this be done in two steps instead of one?
  - “3 steps per user” vs “some number of steps per user”
  - Sometimes we don’t care…sometimes we do

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- Decouple algorithm from infrastructure/implementation
  - Asymptotic notation…?

And now a brief interlude…

Logarithms (refresher)

Let $a, b, c, n > 0$

Exponent Rule:	$\log(n^a) = a \log(n)$
Product Rule:	$\log(an) = \log(a) + \log(n)$
Division Rule:	$\log(n/a) = \log(n) - \log(a)$
Change of Base:	$\log_b(n) = \log_c(n) / \log_b(n)$
Log/Exponent are Inverses:	$b^{\log(n)} = \log_b(b^n) = n$

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Log/Exponent are Inverses: $b^{\log(n)} = \log_b(b^n) = n$

In this class, always assume log base 2 unless specified otherwise

Now back to “fast”...

Attempt #2: Growth Functions

Not a function in code…but a mathematical function:

$$f(n)$$

n: The “size” of the input

ie: number of users,rows, pixels, etc

f(n): The number of “steps” taken for input of size n

ie: 20 steps per user, where n = |Users|, is 20 x n

Some Basic Assumptions:

Problem sizes are non-negative integers

$$n \in \mathbb{Z}^+ \cup \{0\} = \{0, 1, 2, 3, ...\}$$

We can’t reverse time…(obviously)

$$f(n) > 0$$

Smaller problems aren’t harder than bigger problems

$$\forall n_1 < n_2, f(n_1) \leq f(n_2)$$

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$$f : \mathbb{Z}^+ \cup \{0\} \to \mathbb{R}^+$$

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First Problem...

We are still implementation dependent

$$f_1(n) = 20n$$
$$f_2(n) = 19n$$

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Does 1 extra step per element really matter...?

First Problem...

We are still implementation dependent

$$f_1(n) = 20n$$

$$f_2(n) = 19n$$

$$f_3(n) = \frac{n^2}{2}$$

f_1 and f_2 are much more "similar" to each other than they are to f_3

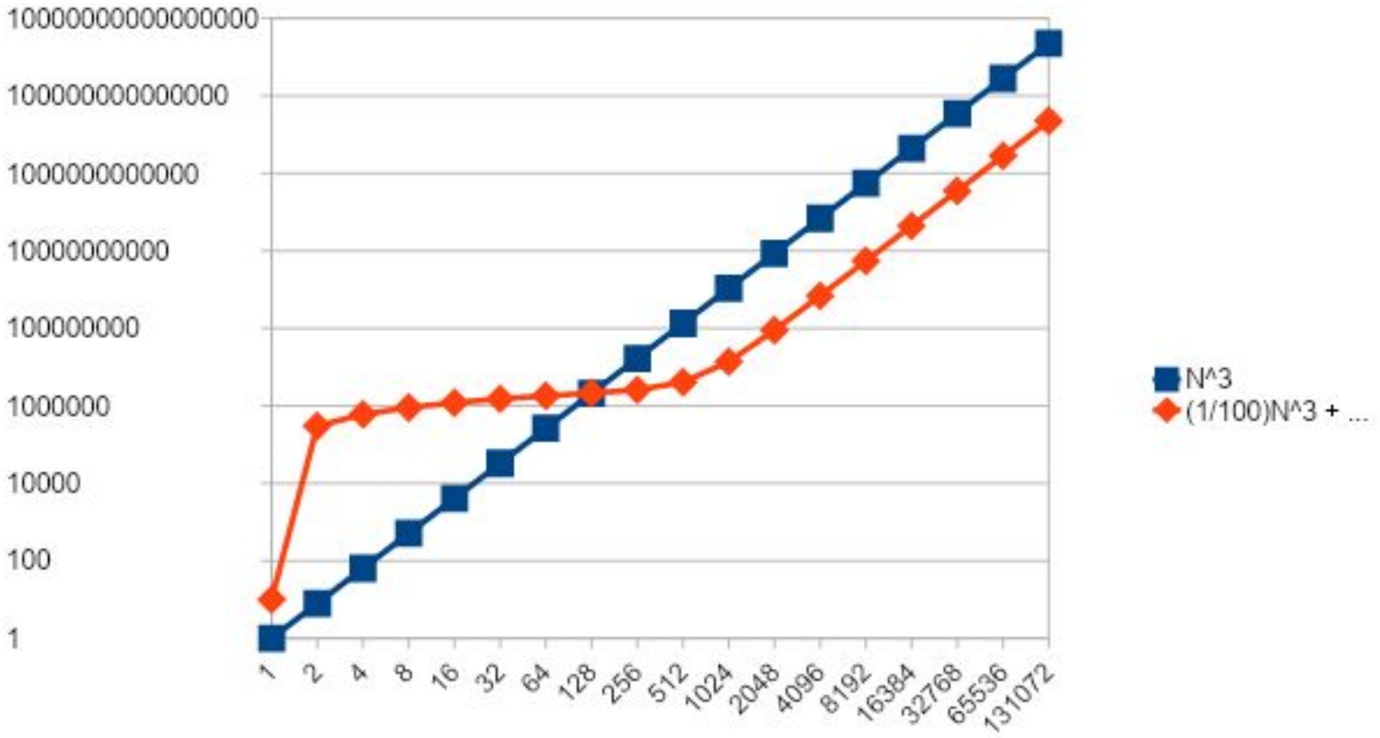
How Do We Capture Behavior at Scale?

Consider the following two functions:

$$\frac{1}{100}n^3 + 10n + 1000000\log(n)$$

$$n^3$$

How Do We Capture Behavior at Scale?



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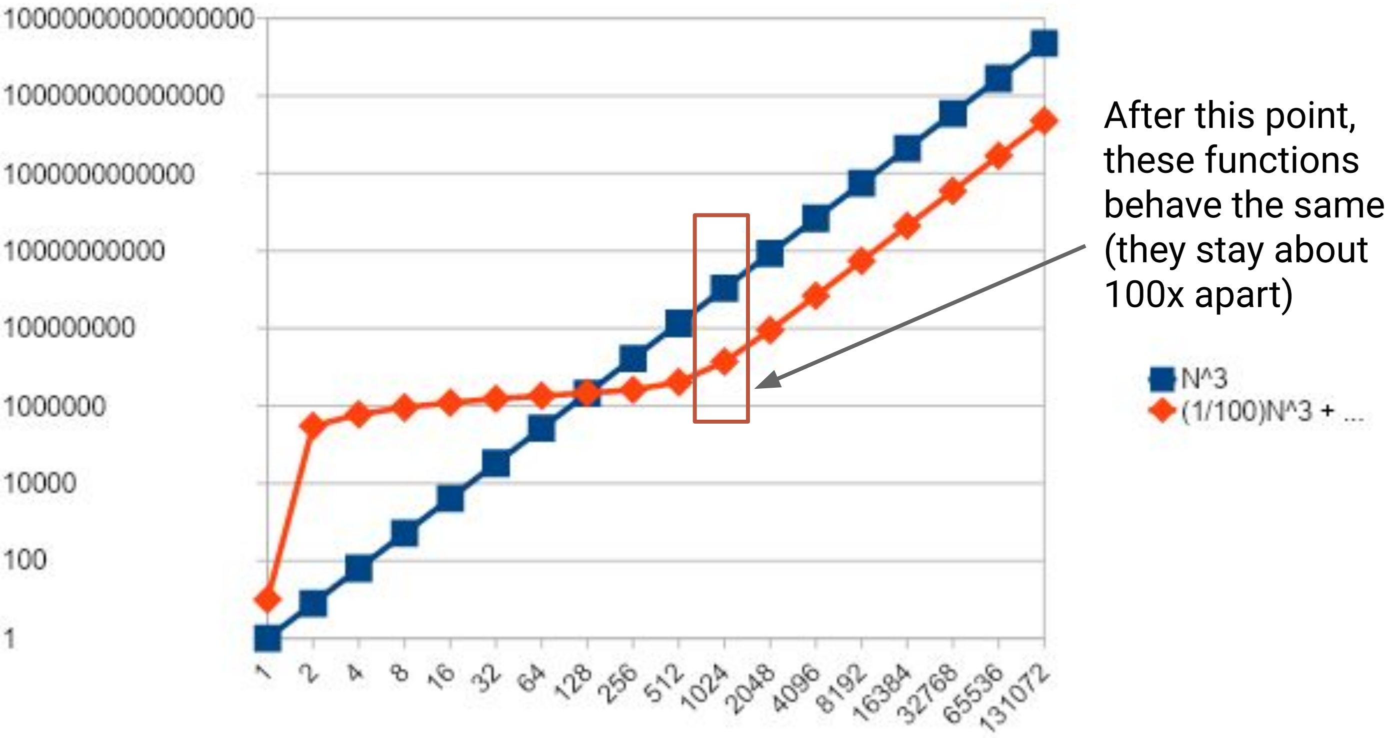

After this point, these functions behave the same (they stay about 100x apart)

How Do We Capture Behavior at Scale?

$$\lim_{n \to \infty} \frac{\frac{1}{100}n^3 + 10n + 1000000\log(n)}{n^3}$$

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$$= \lim_{n\to\infty} \frac{1}{100} + \boxed{\lim_{n\to\infty} \frac{10}{n^2}} + \boxed{\lim_{n\to\infty} \frac{1000000\log(n)}{n^3}}$$

These terms go to 0

How Do We Capture Behavior at Scale?

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$$= \frac{1}{100}$$

Attempt #3: Asymptotic Analysis

Consider two functions, f(n) and g(n)

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty$$

In this particular case, f grows w.r.t. n faster than g

So…if f(n) and g(n) are the number of steps two different algorithms take on a problem of size n, which is better?

Attempt #3: Asymptotic Analysis

Case 1: $\lim_{n\to\infty}\frac{f(n)}{g(n)} = \infty$ *(f grows faster; g is better)*

Case 2: $\lim_{n\to\infty}\frac{f(n)}{g(n)} = 0$ *(g grows faster; f is better)*

Case 3: $\lim_{n\to\infty}\frac{f(n)}{g(n)} = some\ \ constant$ *(f and g "behave" the same)*

Goal of “Asymptotic Analysis”

We want to organize runtimes (growth functions) into different ***Complexity Classes***

Within the same complexity class, runtimes “behave the same”

Goal of "Asymptotic Analysis"

"Strategic Optimization" focuses on improving the complexity class of your code!

Back to Our Previous Example...

$$\frac{1}{100}n^3 + 10n + 1000000\log(n)$$

The 10n and 1000000 log(n) “don’t matter”

The 1/100 “does not matter”

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$$\frac{1}{100}n^3 + 10n + 1000000\log(n)$$

The 10n and 1000000 log(n) “don’t matter”

The 1/100 “does not matter”

n^3 is the dominant term, and that determines the “behavior”

Why Focus on Dominating Terms?

$f(n)$	**10**	**20**	**50**	**100**	**1000**
$log(log(n))$	0.43 ns	0.52 ns	0.62 ns	0.68 ns	0.82 ns
$log(n)$	0.83 ns	1.01 ns	1.41 ns	1.66 ns	2.49 ns
n	2.5 ns	5 ns	12.5 ns	25 ns	0.25 µs
$nlog(n)$	8.3 ns	22 ns	71 ns	0.17 µs	2.49 µs
n^2	25 ns	0.1 µs	0.63 µs	2.5 µs	0.25 ms
n^5	25 µs	0.8 ms	78 ms	2.5 s	**2.9 days**
2^n	0.25 µs	0.26 ms	**3.26 days**	**10^{13} years**	**10^{284} years**
$n!$	0.91 ms	**19 years**	**10^{47} years**	**10^{141} years**	🤯

Why Focus on Dominating Terms?

$$2^n \gg n^c \gg n \gg log(n) \gg c$$